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The Finite Element Method and the Solution of Some Geophysical Problems

O. C. Zienkiewicz

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The finite element method and the solution of some geophysical problems

By O. C. Zienkiewicz
Department of Civil Engineering, University of Wales, Swansea

The finite element method has become established as a powerful tool for the solution of many problems of continuum mechanics where its physical interpretation, by analogy with discrete problems of structural analysis permits the user to exercise a considerable degree of insight and judgement in its use. Further it is now a recognized mathematical procedure of approximation which embraces many older methodologies (such as the finite difference method) as a subclass.

In the field of geological studies its impact is fairly recent and only a limited application has been made to date. The techniques used here have been limited to those established over a decade ago in the parallel fields and recent developments and possibilities barely touched upon. In this paper the author therefore attempts to

(*a*) outline some of the general mathematical and practical aspects of the method with illustrations from various fields which are relevant to geological problems,

(*b*) survey accomplishments already made in geology and geotechnical fields, and

(*c*) suggest some possible new extensions of application.

Introduction

In the study of geological problems, if more than purely descriptive and intuitive hypotheses are to be established, a need exists for the establishment of a suitable physical-mathematical model and the subsequent solution of the equations presented. In this respect the science of geology and geophysics does not differ from other branches of science and engineering where the same logical procedures are followed. Indeed the mechanistic models the engineer designs for his study of stresses and strain in structures or for viscous and plastic flow of substances he is forming, must be extremely similar in kind if not in scale to those pertinent in geology. Undoubtedly much benefit can accrue from an exchange of information. It is in this capacity that I, as an engineer, am addressing this body of geologists with whose science my own profession of Civil Engineering is intimately connected.

In the problem modelling – and solution sequence the greatest difficulties are usually presented in the second stage. Here except for a few trivial configurations, analytical, closed form solutions are not possible and alternative numerical solutions must be sought. Finite difference methods have been here the first to be widely applied and it was a member of this Society and an engineer, Sir Richard V. Southwell (1956) who made such solutions practically possible for the first time by his 'relaxation method'. The subsequent advent of the computer has made relaxation methods obsolete but nevertheless finite difference approximations, which were a part of Southwell's general scheme, have continued to flourish to the present day. In the mid-fifties engineers have introduced an alternative process of approximation, in many ways more 'in-tune' with the computer and named it the 'finite element' method (Turner *et al.* 1956; Zienkiewicz 1971; Strang & Fix 1973). Although initially constrained to problems of structural engineering the process was soon discovered to be one of general applicability to all properly formulated models of continua. Today, rediscovered by mathematicians, it has become an almost standard method for the solution of many engineering and physical problems and has been shown to include the finite difference procedures as its 'subset' Zienkiewicz (1972). Table 1 gives a brief family tree of the method.

Clearly the application of the finite element method to geological situations is a natural development. What I shall attempt to do in this paper is to summarize the essence of the finite element method as conceived today and show how some technical detail improvements and applications already made to various physical problems of engineering can be turned directly to the gelogical benefit. Here perhaps it is important to stress that the use of finite element methodology requires often a quite sophisticated programming effort and if suitable 'analogues' can be chosen by the geologist from either branches of science or engineering he can avoid much tedious development work and concentrate his efforts on the model and on the results – where properly his expertise belongs.

Table 1. Family tree of finite element methods

ENGINEERING		MATHEMATICS: trial functions		MATHEMATICS: finite differences
		variational methods	weighted residuals	
		Rayleigh 1870 Ritz 1909 ↓	Gauss 1795 Galerkin 1915 Biezeno-Koch 1923	Richardson 1910 Liebman 1918 Southwell 1940 ↓
structural analogue substitution	direct continuum elements	piecewise continuous trial functions		variational finite differences
Hrenikoff McHenry 1943 Newmark 1943	↓ Argyris 1955 Turner/Clough 1956	Courant 1943 Prager/Synge 1947 ↓	↓	Varga 1962
⟶		PRESENT DAY FINITE ELEMENT METHOD		⟵

It is important however for him to have some understanding of the procedures and if this paper helps him here in some measure, its objectives will be achieved.

Already much application of the finite element procedure to problems of pure geology and to the geophysical situations on the fringe between engineering and geology (soil/rock mechanics) have been made. On the former side the notable efforts of Dieterich (Dieterich & Onat 1969; Dieterich & Carter 1969), Voight & Samuelson (1969), Stephansson & Berner (1970), Douglas (1970), Hudleston & Stephansson (1973), Bott & Dean (1972) and Service (1973) must be mentioned. In soil and rock mechanics the applications are so numerous that no specific references will be made here but a bibliography in Zienkiewicz (1971) will suffice to give an overall idea. What is notable in the above mentioned application of the finite element process to geology is that they deal either with linear elastic or linear viscous approximations to the solid mechanics problems. Nonlinear material behaviour assumptions and applications beyond those of elasticity (or its simple analogue of slow viscous flow) are still absent. Important problems of convective flows, thermal diffusion, etc. are still dealt with using the finite difference techniques. This perhaps follows too closely the path of development which occurred in engineering where the simple linear structural problem was given by far the greatest share of attention before the parallel and equally important application in other fields were made.

To reverse the process in the next section dealing with the basis of the method, the heat diffusion equation will be used as the basic example – and other applications will be implied.

The basis of the finite element method

The physical-mathematical model of a continuum problem is most generally presented in the form of a differential equation $A(\phi) = 0$ and if its associated boundary conditions $B(\phi) = 0$ to be satisfied within a problem domain Ω and on its boundaries Γ and where the unknown sought is ϕ.

Thus for instance the problem of steady state convective – conductive heat transfer in a two dimensional case with prescribed boundary temperatures can be defined as

$$A(\phi) = \frac{\partial}{\partial x}\left(k\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial \phi}{\partial y}\right)+u\frac{\partial \phi}{\partial \lambda}+v\frac{\partial \phi}{\partial x}+Q = 0 \tag{1}$$

in Ω with

$$B(\phi) = \phi-\bar{\phi} = 0 \tag{2}$$

in Γ where $\bar{\phi}$ are known, prescribed, values of the temperature ϕ. Here k, u, v, Q are respectively conductivity coefficients, velocities and heat generation terms which may or may not be temperature dependent.

In the finite element process the objective is to obtain *an approximation* which will determine the unknown function ϕ by a finite number of unknown parameters $\boldsymbol{a}$. Further, the algebraic equations from which these parameters are to be determined must satisfy the basic conditions that the contribution to these equations are obtained by a simple addition of contributions obtained from subdomains Ω^e which we shall call 'finite elements'. We shall expect that

$$\Sigma \Omega^e = \Omega, \tag{3}$$

i.e. that the 'whole' is simply a sum of its 'parts' and that the parts are of a simple geometrical shape so that the contributions can be evaluated in a *standard*, *repeatable form*.

To obtain such approximation it is necessary

(*a*) to expand the unknown function ϕ in an approximate form as

$$\tilde{\phi} = \sum_{i=1}^{n} N_i a_i + \phi_0 \quad N_i = N_i(x, y) \tag{4}$$

where N_i are suitable *shape* or *trial* functions and ϕ_0 satisfies the boundary conditions (2);† and

(*b*) to form the approximating equations in an integral form writing

$$\int_\Omega W_j A(\tilde{\phi})\mathrm{d}\Omega = 0 \quad j = 1-n. \tag{5}$$

Clearly for any integrable function F

$$\int_\Omega F\mathrm{d}\Omega = \Sigma\int_{\Omega^e} F\mathrm{d}\Omega \tag{6}$$

and therefore the additivity of element contributions is obtained in the equations of form (5).

† Generally we write simply

$$\phi_0 = \sum_{i=n+1}^{m} N_i a_i$$

to preserve a standard form and specify *a priori* some values of a_i.

Specializing above to the particular case considered and applying Green's theorem we have (for weighting or test functions W_j such that $W_j = 0$ on Γ)

$$\int_\Omega W_j \left(\frac{\partial}{\partial x}\left(k\frac{\partial\tilde{\phi}}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial\tilde{\phi}}{\partial y}\right)+u\frac{\partial\tilde{\phi}}{\partial x}+v\frac{\partial\tilde{\phi}}{\partial y}+Q\right)\mathrm{d}\Omega$$

$$\equiv -\int\left(\frac{\partial W_i}{\partial x}k\frac{\partial\tilde{\phi}}{\partial x}+\frac{\partial W_j}{\partial x}k\frac{\partial\tilde{\phi}}{\partial x}-W_j u\frac{\partial\tilde{\phi}}{\partial x}-W_j v\frac{\partial\tilde{\phi}}{\partial y}-Q\right)\mathrm{d}\Omega = 0. \tag{7}$$

This gives a system of equations written in a matrix form as

$$\boldsymbol{Ka}-\boldsymbol{f} = 0 \tag{8}$$

with

$$K_{ij} = \int_\Omega \left(\frac{\partial W_j}{\partial \alpha}k\frac{\partial N_i}{\partial x}+\frac{\partial W_j}{\partial y}k\frac{\partial N_i}{\partial y}-W_j u\frac{\partial N_i}{\partial x}-W_j v\frac{\partial N_i}{\partial y}\right)\mathrm{d}\Omega$$

$$f_i = \int_\Omega W_j\, Q\, \mathrm{d}\Omega \tag{9}$$

which allows the parameters a to be found by algebraic solutions. Clearly

$$K_{ij} = \Sigma K^e_{ij}, \quad f_i = \Sigma f^e_i \tag{10}$$

by virtue of the integration properties and typical contributions are readily found for each element.

The procedure described above will readily be recognized as one of the classical trial function approximations of the type dating back to Rayleigh, Ritz, Galerkin and others but the essential feature of the finite element process is the use of *locally defined trial and weighting functions* which are associated with the element. Similarly the parameters a_i take up conveniently (but by no means always) the values of the unknown function ϕ at nodes $(x_i\ y_i)$. With these provisos the calculations can be stereotyped to any element and the recipe for forming the base equation is immediately available.

As the locally defined functions are simply equal to zero for most of the elements the approximating equation will in general be 'banded' (sparse) thus facilitating solution.

Although the choice of weighting (test) functions is quite arbitrary it is convenient to use a Galerkin form with

$$W_j = N_j. \tag{11}$$

The problem is now formulated *approximately* and its numerical solution can be readily obtained by solving equations (8). If the coefficients k, Q are independent of ϕ then equations are linear and a direct solution approach may be used – if not one of the many available iterative approaches has to be adopted to obtain the numerical solution.

It must always be remembered that, just as in finite difference calculus, the solution is only an approximate one. The degree of approximation can be established by convergence studies (reduction of the element subdivision) or comparison with occasionally available exact solutions. The 'goodness' of the approximation depends very clearly on the choice of the trial function and element forms. In some realistic problems excellent approximation can be achieved with 50–100 unknown parameters while in others 3–10000 or more unknowns have been involved. Locally based shape functions for some typical two dimensional elements are shown in figure 1, and in general those with more nodes – using a higher order expansion – are more accurate and economic. The simple triangle which was one of the original hallmarks of the finite element method is today largely superseded.

Much development in basic mathematical theory has gone into the process of efficient formulation, convergence study and solution methodology of the finite element method. Several thousand papers and a dozen or so texts are now in existence – so obviously this 'concise' finite element story omits much of importance. Nevertheless it is hoped that this basic and the simple example shows the essentials which are applicable to almost any form of continuum model.

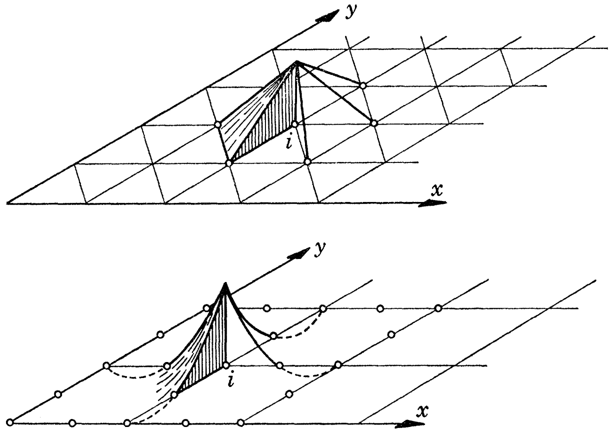
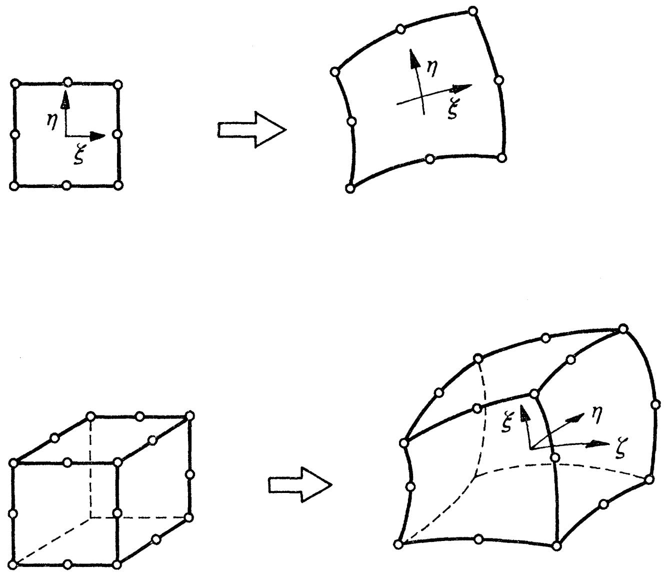


FIGURE 1. Typical 'locally based' shape functions for finite element analysis: (*a*) local linear functions associated with a node *i* of a triangulated field; (*b*) local quadratic functions associated with a node *i* of a quadrilateral field.



FIGURES 2 and 3. Elements with curvilinear mapping: 2 (top), isoparametric 2-dimensional element. 3 (bottom), isoparametric 3-dimensional element.

The equation of elasticity, plasticity or viscous flow could be treated by precisely the same general process and the interested reader is referred to texts on the subject. He will find that one of the most popular and efficient elements used is an isoparametric parabolic element – shown in figures 2 and 3 in its two and three dimensional forms, especially if certain 'smoothing' relationships are used in conjunction with numerical integration necessary for the calculation of its properties such as defined in equations (9) and (10). In most of the subsequent examples this element will be amply demonstrated.

Some problems of geophysics amenable to finite element solution

The materials of the Earth's surface and interior range widely from what we would describe for short term loading purposes as solids in the cool crust to more or less viscous fluid in the mantle and core. Indeed the separation of solid and fluid state is qualitative rather than quantitative and, as Marcus Reiner, the father of rheology, once remarked, 'everything flows' especially if geological time scales are considered.

If we describe the strains by a quantity $\boldsymbol{\varepsilon}$† and the corresponding stresses by a quantity $\boldsymbol{\sigma}$ the instantaneous 'solid', elastic response gives a functional relation of the form

$$\boldsymbol{\varepsilon}_e = \boldsymbol{\varepsilon}_e(T, \boldsymbol{\sigma}) \tag{12}$$

where T is the temperature (or other state variable). For a fluid on the other hand the strain rate is related to stresses as

$$\mathrm{d}/\mathrm{d}t\, \epsilon_f \equiv \dot{\boldsymbol{\varepsilon}}_f = \dot{\boldsymbol{\varepsilon}}_f(T, \sigma). \tag{13}$$

In real materials both elastic solid and fluid charactersitics exist and strains are defined as a sum of both components

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_f \tag{14}$$

and in a general form the above three equations are capable of describing the constitutive relations of elastic, elasto-visco-plastic, and visco-plastic behaviour forms. Linearized forms of above relationships

$$\left.\begin{aligned} \boldsymbol{\varepsilon}_e &= \boldsymbol{D}(T)\boldsymbol{\sigma} \\ \dot{\boldsymbol{\varepsilon}}_f &= \bar{D}(T)\boldsymbol{\sigma} \end{aligned}\right\} \tag{15}$$

are often used to describe linear elastic solid or 'Newtonian' fluid behaviour.

To solve a particular problem for strains, stresses, velocities and other quantities, the equilibrium and compatibility conditions have to be satisfied and in general the temperature T has to be determined. Clearly the temperature distribution problem has to be solved simultaneously, as, if this is typified by equation (1) we note that it is coupled by the velocity values and by the heat generation term Q in which mechanical work dissipation terms enters. The complete coupled solution, while possible, is computationally very demanding and it is desirable to make simplifying assumptions for various categories of problems. In particular when deformation becomes large the elastic component $\boldsymbol{\varepsilon}_e$ of strain can often be neglected and pure flow needs only to be considered. This means that actual displacements do not enter the problem and that attention can be *focused on velocities* alone. Further in some problems the coupling is almost non-existent and a separate solution of the velocity/thermal equilibrium studies can be accomplished. Thus we can broadly distinguish three categories of approximation which have their engineering counterparts.

† For simplicity we use a vector-matrix type of notation. For those preferring the more standard tensor notation the definitions are

$$\boldsymbol{\varepsilon}^T = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{23}, \epsilon_{31}], \text{ etc.}$$

(A) '*Displacement*' *problems with predominant elastic behaviour; uncoupled or isothermal behaviour*

These geologically correspond to *local, short term events of the Earth's crust*. Typical problems here range from static *stability* and *deformation problems of embankments; rock cavities or even mountain ranges* to the *mechanism of earthquake wave propagation*. Further *fracture or fault development* studies lie in this category.

In such problems the thermal conditions may be obtained by an uncoupled solution (as due to negligible velocities and short time considerations, convective terms and heat generation can be entirely omitted).

(B) '*Velocity*' (*flow*) *problems with uncoupled thermal behaviour*

Here with longer time scale involved it is reasonable to assume a negligible elastic response and continuous flow is taken to occur. Once again, however, it is assumed that rates of flow are sufficiently small for a neglect of convective and heat generation terms and a consequent uncoupling of the thermal equations. Typical here are problems of fold development or simplified studies of local flows of edges of crustal 'plates'.

In this uncoupled range complex problems of the thermal kind may have to be solved involving phase changes. The classical problem of Stefan, i.e. cap melting and its more interesting counterparts in rock are typical examples.

(C) *Velocity* (*flow*) *problems with coupled thermal behaviour*

This is the most ambitious generalization yet attempted and undoubtedly the one which will yield some understanding of general geological movements occurring in the mantle and Earth's interior.

Coupling of temperature (and indeed state and chemical) development occurs now via the presence of convection and heat generation due to flow. In the flow problem density changes due to phase and temperature changes as well as viscosity dependence on thermal properties become the most significant features. Clearly, the answers to many problems of convective currents originating large scale crust movements will ultimately be given by such solutions!

This broad classification could further be subdivided into (*a*) linear, (*b*) non-linear approximations. Clearly linearization is but a simplifying assumption introduced to ease computation.

What then is the status of the finite element method vis-à-vis these classes of problems and how much has already been achieved?

The state of the 'art' in engineering applications of relevant kind

Solid mechanics – category A (*a*)–(*b*)

This is probably the most 'worked at' and developed area of engineering activity and solutions of problems with and without the inclusions of time dependent terms have reached the status of being a 'standard procedure' in engineering applications. 'Off the shelf' programs are now widely available for the linear solution and can be obtained also for nonlinear phases. On the fringes of geology stand here actual application to flow of soil and rock in foundations and one such recent application is illustrated in figure 4 where study of an idealized embankment is

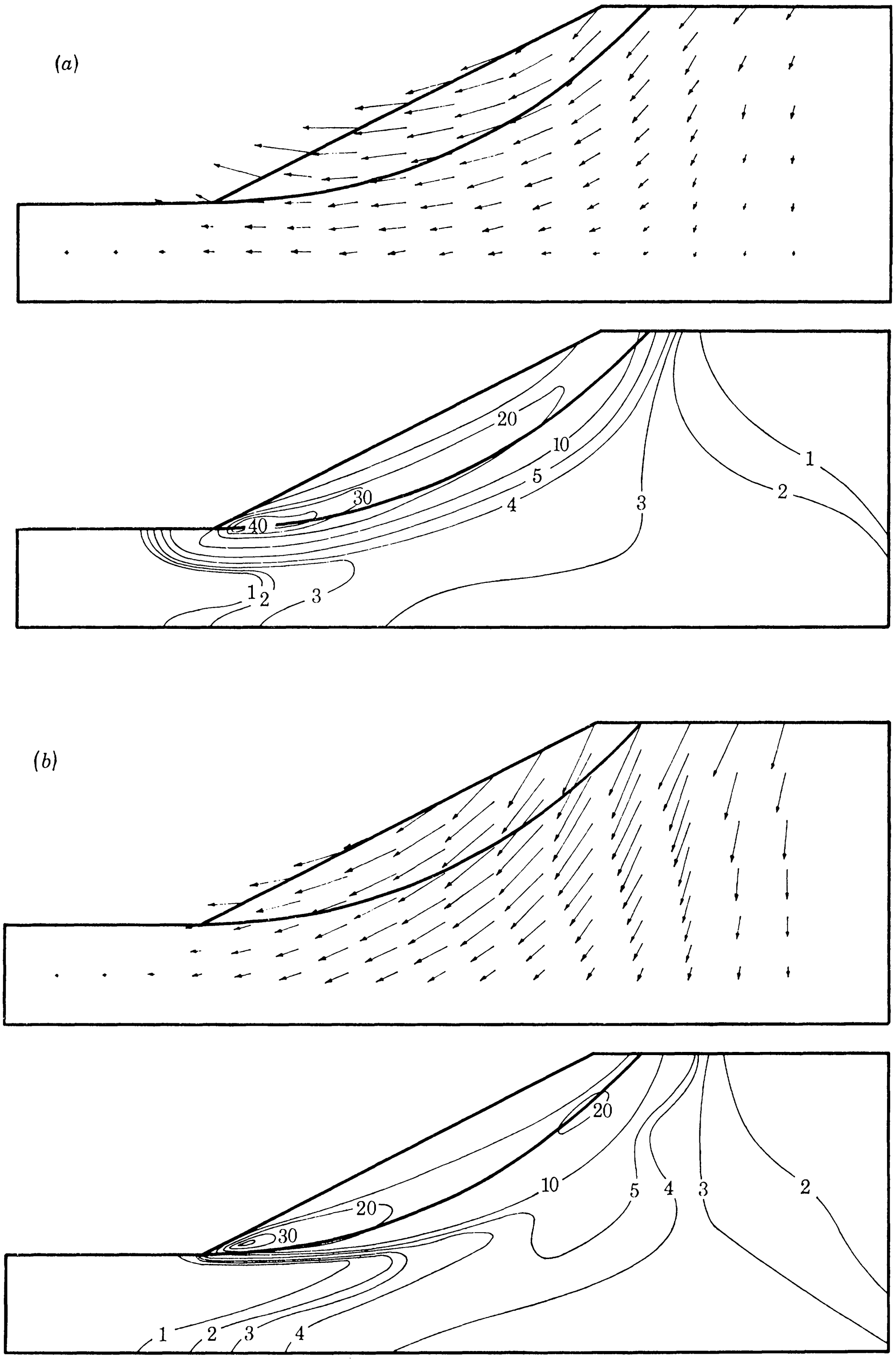


FIGURE 4. Study of elasto-visco-plastic collapse in an embankment associated and non-associated flow rules. Flow velocity patterns and contours of maximum deviatoric strain rate. (a) Associative ($\theta = \phi = 20$); (b) non-associative ($\theta = 0$, $\phi = 20$).

made. Here the material is assumed to have a behaviour similar to that of a Bingham-fluid (Zienkiewicz & Cormeau 1974; Zienkiewicz, Humpheson & Lewis 1976) with

$$\boldsymbol{\varepsilon} = \boldsymbol{D\sigma} + \boldsymbol{\varepsilon}_{\mathrm{f}} \tag{16}$$

where $\dot{\boldsymbol{\varepsilon}}_{\mathrm{f}}$ is given by a highly nonlinear expression in terms of stress components $\boldsymbol{\sigma}$.

Similar nonlinear models have been used for studies of cavities in salt domes and their closures (Anderson 1974).

In purely geological literature so far only linearized models have been used and such applications have been given in studies of 'boudins' (Stephansson & Berner 1970) and continental margins (Bott & Dean 1972). Clearly a direct borrowing from the engineering literature will permit suitably identified problems to be dealt with simply and no new program development is required.

The separation of engineering and geology at this front is very narrow – and one may well ask whether such semi-natural phenomena as the slip of the Toc mountain into the Vajont reservoir in 1961 fall into the category of geophysics or engineering.

Viscous flow problems – category B

The formulation and solution of flow problems is newer in the context of the finite element field and practically significant solutions have only recently begun to emerge. Indeed here the finite difference methodology has been more extensively developed and as yet is not superseded. A full review of the problems is given in a recent conference held at Swansea (Oden *et al.* 1974). While difficulties still exist at high speed flow (high Reynolds number) the geological flow phenomena fall fortunately into the 'creeping' category where convective acceleration terms are small (or negligible). For such situations the finite element process is ideally suited and has already shown its superiority over other methods. Indeed it is possible to establish a direct analogy with elastic behaviour of solids and once again to borrow directly both the method and the program for it.

Dieterich & Onat (1969) have been some of the first to discuss this analogy (also Dieterich & Carter 1969) and Voight & Samuelson (1969), Stephansson & Berner (1970) and Hudleston & Stephansson (1973) have successfully used it in the treatment of slow viscous flow resulting in the formation of folds. In all the applications made to date here only simple linear problems of constant viscosity were solved; however, the method permitted a follow through of gross deformation by a simple 'updating of coordinates'. In engineering literature similar applications have also been made but with both constant and variable viscosities. In particular much attention has been given to the highly non-Newtonian behaviour of materials such as steel where plastic flow occurs (Bingham type behaviour) and in general the viscosity can be written as a function of both temperature and strain rate (Zienkiewicz & Godbole 1974, 1975; Cornfield & Johnson 1973) or

$$\bar{D} = \bar{D}(T, \dot{\varepsilon}). \tag{17}$$

Even with very large variation of viscosity, equations can be solved in a few iterations and quite dramatic results obtained.

Clearly some nonlinear behaviour is present to a very large extent in rocks where the viscosity will be dependent on all the stress (or strain rate) invariants. To show some of the possibilities in figure 5 we solve – using now a quite small number of isoparametric quadratic elements –

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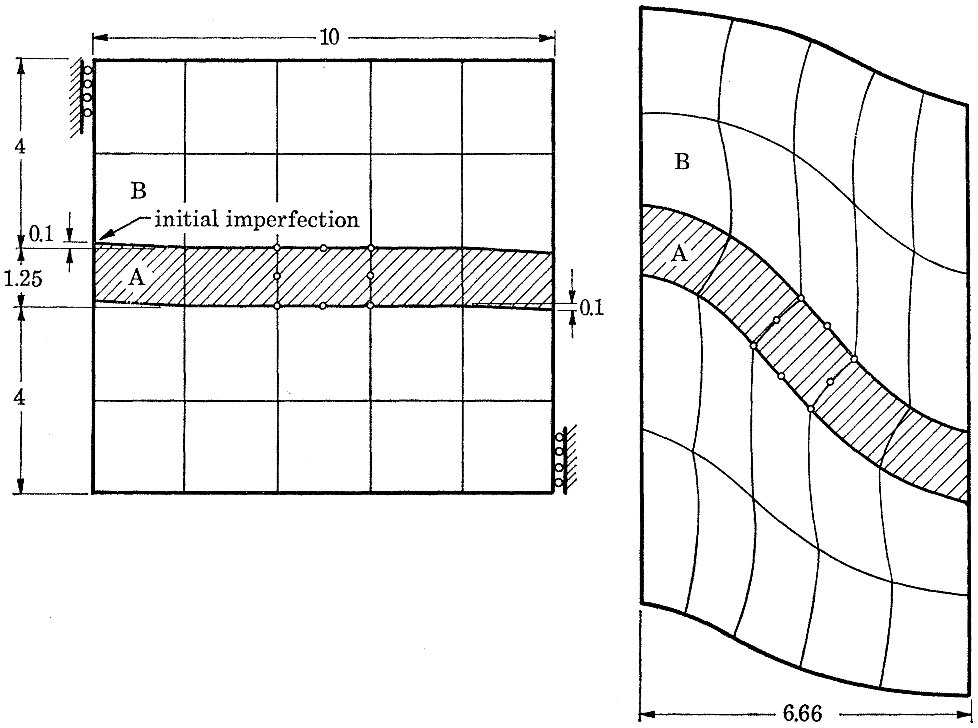
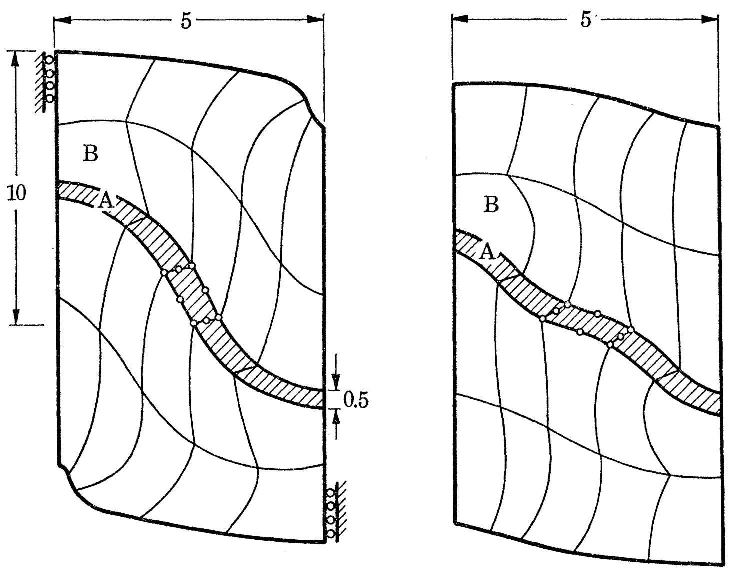


FIGURE 5. A hypothetical folding process. Linear viscosity. $\mu_B/\mu_A = 1/100$; 30 % shortening.



$\sigma_y^B/\sigma_y^A = 1/1000$ $\sigma_y^B/\sigma_y^A = 1/100$

FIGURE 6. A hypothetical folding process. Plastic (Bingham) flow. 50 % shortening.

a typical problem of folding using a linear viscosity and in figure 6 we insert a purely plastic behaviour for comparison purposes computed by J. Williams, Swansea. In both studies the fold is not assumed but a simple perturbation of symmetry is imposed.

The possibilities offered are large and doubtless application of such procedures may throw further light on fold formation and possibly fracturing.

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Uncoupled – phase change thermal problems

Thermal diffusion equations (with or without convective terms) again one of the standard problems of solution by finite element procedures. Once again programs are quite widely available and directly applicable to many geological situations. Phase change problems associated with such thermal conduction are still however a serious computational hazard and indeed only last year a conference on various mathematical procedures capable of dealing with these has been held at Oxford (Ockendon & Hodkgins 1974).

However, even in this class of problem the finite element procedure has been found to be readily adaptable and transient solutions now can be obtained. In figure 7 we show a recent computation of the progress of a freezing front in ground due to refrigeration necessary to stabilize it (Comini *et al.* 1974).

It would appear to be of interest to follow such 'freezing' and similar chemical changes occurring in the Earth's mantle by identical processes.

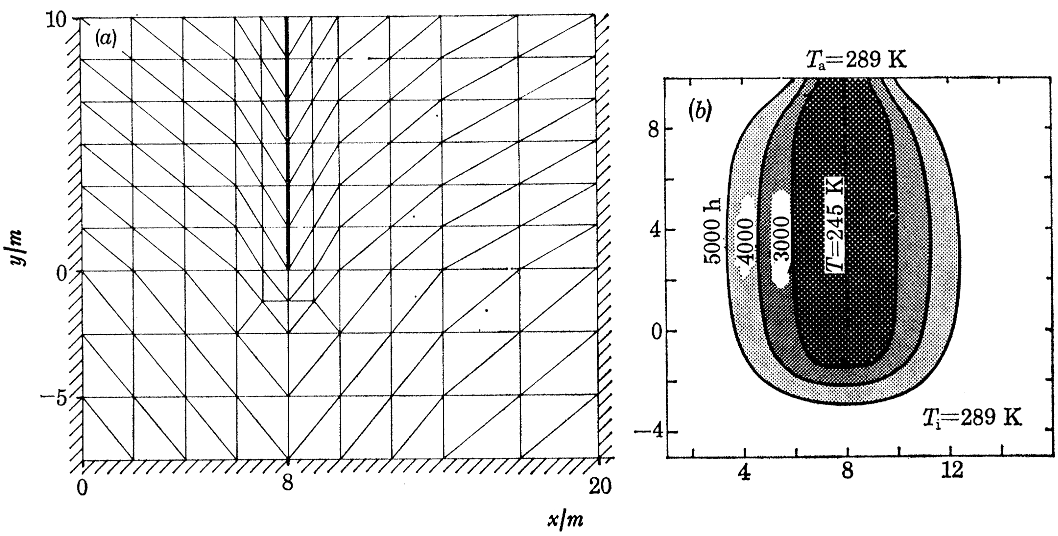

FIGURE 7. Progression of a freezing front through soil due to artificial cooling.

Coupled thermal flow problems – category C

Here the finite element method and indeed its competitors have not yet succeeded in obtaining a significant number of solutions but recent research shows that even in highly coupled situations possibilities of solution exist. Kawahara (1976) and Zienkiewicz, Gallagher & Hood (1975) have formulated and solved typical problems of density generated currents due to thermal interaction in a fluid. A typical solution is shown in figure 8. While such solutions are still in the research phase and programs are not yet widely available – they would appear to offer a possibility of solving significant geological problems. A scan through the literature showing the possible mechanisms occurring in gravitational instability of intrusions or general convective current distributions on global scale appear to be subjects now ripe for a quantitative analysis.

Problems of uncoupled heat flow in the sinking lithosphere near the island area of Japan have already been attempted (McKenzie 1970; Hasebe, Fujii & Veda 1970). It appears that such

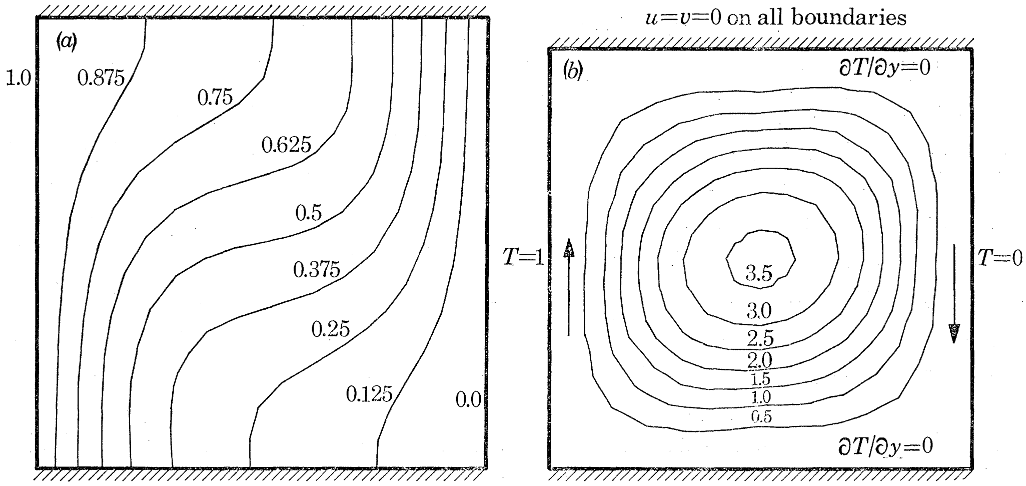

FIGURE 8. Density generated currents in a cavity. (*a*) Temperatures RA = 5000, PR = 1000 heated enclosure. (*b*) Streamlines as (*a*), 16 elements (cubic u, parabolic p) $p = 0$ at centre.

a 'local' study could be dealt with efficiently by numerical procedure already developed in the finite element field. Some dramatic applications to diapirism have already been made by Berner, Ramberg & Stephansson (1973) and doubtless more work will continue with more sophisticated models of behaviour.

CONCLUDING REMARKS

In this brief survey we have touched on some problems already solved and on others yet requiring solution for which the finite element process is manifestly suitable. The short space does not permit to enlarge the discussion to multiphase geological problems such as arise in oil reservoir engineering and movement of oil/water interfaces and their coupling with ground stresses. Indeed many important feasible uses of the general mathematical methodology will be opened up undoubtedly as the need arises. As an engineer I would like to offer to the geologists the tools and our collaboration when the need for it occurs.

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